

UNIVERSAL GRÖBNER BASES FOR MAXIMAL MINORS

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ABSTRACT. Bernstein, Sturmfels and Zelevinsky proved in 1993 that the maximal minors of a matrix of variables form a universal Gröbner basis. We present a very short proof of this result, along with broad generalization to matrices with multi homogeneous structures. Our main tool is a rigidity statement for radical Borel fixed ideals in multigraded polynomial rings.

INTRODUCTION

A set G of polynomials in a polynomial ring S over a field is said to be a universal Gröbner basis if G is a Gröbner basis with respect to every term order on S . Twenty years ago Bernstein, Sturmfels and Zelevinsky proved in [3, 12] that the set of the maximal minors of an $m \times n$ matrix of variables X is a universal Gröbner basis. Indeed, in [12] the assertion is proved for certain values of m, n and the general problem is reduced to a combinatorial statement that it is then proved in [3]. Kalinin gave in [10] a different proof of this result. Boocher proved in [4] that any initial ideal of the ideal $I_m(X)$ of maximal minors of X has a linear resolution (or, equivalently in this case, defines a Cohen-Macaulay ring).

The goal of this paper is twofold. First, we give a quick proof of the results mentioned above. Our proof is based on a specialization argument, see Section 1. Second, we show that similar statements hold in a more general setting, for matrices of linear forms satisfying certain homogeneity conditions. More precisely, in Section 3 we show that the set of the maximal minors of an $m \times n$ matrix $L = (L_{ij})$ of linear forms is a universal Gröbner basis, provided that L is column-graded. By this we mean that the entries L_{ij} belong to a polynomial ring with a standard \mathbb{Z}^n -graded structure, and that $\deg L_{ij} = e_j \in \mathbb{Z}^n$. Under the same assumption we show that every initial ideal of $I_m(L)$ has a linear resolution. Furthermore the projective dimension of $I_m(L)$ and of its initial ideals is $n - m$, unless $I_m(L) = 0$ or a column of L is identically 0 (notice that, under these assumptions, the codimension of $I_m(L)$ can be smaller than $n - m + 1$).

If instead L is row-graded, i.e. $\deg L_{ij} = e_i \in \mathbb{Z}^m$, then we prove in Section 4 that $I_m(L)$ has a universal Gröbner basis of elements of degree m and that every initial ideal of $I_m(L)$ has a linear resolution, provided that $I_m(L)$ has the expected codimension. Notice that in the row-graded case the maximal minors do not form

Date: February 26, 2013.

2010 Mathematics Subject Classification. Primary 13C40, 14M12 Secondary 13P10, 05B35.

Key words and phrases. Determinantal ideals, Gröbner bases, matroids.

The first two authors were partially supported by the Italian Ministry of Education, University and Research through the PRIN 2010-11 “Geometria delle Varietà Algebriche”. The third author was partially supported by the Swiss National Science Foundation under grant no. PP00P2_123393.

a universal Gröbner basis in general (since every maximal minor might have the same initial term).

The proofs of the statements in Sections 3 and 4 are based on a rigidity property of radical Borel fixed ideals in a multigraded setting. This property has been observed by Cartwright and Sturmfels in [5] and by Aholt, Thomas, and Sturmfels in [1], in special cases. In a polynomial ring with a standard \mathbb{Z}^m -grading, one can take generic initial ideals with respect to the the product of general linear groups preserving the grading. Such generic initial ideals are Borel fixed. The main theorem of Section 2 asserts that if two Borel fixed ideals I, J have the same Hilbert series and I is radical, then $I = J$. This is the rigidity property that we referred to, and which has very strong consequences. For instance if I is Cohen-Macaulay, radical and Borel fixed, then all the multihomogeneous ideals with the same multigraded Hilbert series are Cohen-Macaulay and radical as well.

Extensive computations performed with CoCoA [6] led to the discovery of the results and examples presented in this paper. We thank Christian Krattenthaler for suggesting the elegant proof of formula (4.2.9). This work was done while the authors were at MSRI for the 2012-13 special year in commutative algebra. We thank the organizers and the MSRI staff members for the invitation and for the warm hospitality.

1. A SIMPLE PROOF OF THE UNIVERSAL GB THEOREM

Let K be a field, $S = K[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates, and let $I_m(X)$ be the ideal generated by the maximal minors of X . The goal of this section is giving a quick proof of the following result of Bernstein, Sturmfels, Zelevinsky [3, 12], and Boocher [4]:

Theorem 1.1. *The set of maximal minors of X is a universal Gröbner basis of $I_m(X)$, i.e., a Gröbner basis of $I_m(X)$ with respect to all the term orders. Furthermore every initial ideal of $I_m(X)$ has the same Betti numbers as $I_m(X)$.*

We need the following “Hilfssatz”:

Lemma 1.2. *Let R be a standard graded K -algebra, let M, N, T be finitely generated graded modules R -modules, and $J = (y_1, \dots, y_s) \subset R$ be a homogeneous ideal. Suppose that:*

- (1) *there exists a surjective graded R -homomorphism $f : T \rightarrow N$.*
- (2) *M and N have the same Hilbert series,*
- (3) *M/JM and T/JT have the same Hilbert series,*
- (4) *y_1, \dots, y_s is M -regular sequence.*

Then f is an isomorphism and y_1, \dots, y_s is a T -regular sequence.

Proof. We denote by $\text{HS}(M, x) \in \mathbb{Q}[[x]][x^{-1}]$ the Hilbert series of a finitely generated graded R -module M . For $i = 1, \dots, s$ set $J_i = (y_1, \dots, y_i)$, $T_i = T/J_i T$, $d_i = \deg(y_i)$ and $g_i(x) = \prod_{j=1}^i (1 - x^{d_j}) \in \mathbb{Q}[x]$. Furthermore set $T_0 = T$ and for $i = 0, \dots, s-1$ denote by K_{i+1} the submodule $\{m \in T_i : y_{i+1}m = 0\}$ of T_i shifted by $-d_{i+1}$. Finally set $K_0 = \text{Ker } f$. For $i \geq 0$ we have an exact complex:

$$0 \rightarrow K_{i+1} \rightarrow T_i(-d_{i+1}) \rightarrow T_i \rightarrow T_{i+1} \rightarrow 0$$

where the middle map is multiplication by y_{i+1} . Additivity of dimensions on exact sequences of vector spaces yields:

$$\mathrm{HS}(T_{i+1}, x) = (1 - x^{d_{i+1}}) \mathrm{HS}(T_i, x) + \mathrm{HS}(K_{i+1}, x)$$

and hence

$$\mathrm{HS}(T_{i+1}, x) = g_{i+1}(x) \mathrm{HS}(T, x) + \sum_{j=1}^{i+1} g_{i+1-j}(x) \mathrm{HS}(K_j, x).$$

Since

$$\mathrm{HS}(T, x) = \mathrm{HS}(N, x) + \mathrm{HS}(K_0, x)$$

we may write

$$\mathrm{HS}(T_{i+1}, x) = g_{i+1}(x) \mathrm{HS}(N, x) + \sum_{j=0}^{i+1} g_{i+1-j}(x) \mathrm{HS}(K_j, x)$$

and in particular, for $i+1 = s$,

$$\mathrm{HS}(T/JT, x) = g_s(x) \mathrm{HS}(N, x) + \sum_{j=0}^s g_{s-j}(x) \mathrm{HS}(K_j, x).$$

Since $\mathrm{HS}(N, x) = \mathrm{HS}(M, x)$ and y_1, \dots, y_s is an M -regular sequence we obtain

$$\mathrm{HS}(T/JT, x) = \mathrm{HS}(M/JM, x) + \sum_{j=0}^s g_{s-j}(x) \mathrm{HS}(K_j, x).$$

Hence, by assumption (3), we have

$$0 = \sum_{j=0}^s g_{s-j}(x) \mathrm{HS}(K_j, x).$$

Since $\mathrm{HS}(K_j, x)$ are powers series with non-negative terms and $g_i(x)$ are polynomials with least degree term coefficient equal to 1, we conclude that $\mathrm{HS}(K_j, x) = 0$ for $j = 0, \dots, s$. Hence $K_j = 0$ for $j = 0, \dots, s$. \square

Proof of Theorem 1.1. We may assume without loss of generality that K is infinite. Let $A = (a_{ij})$ be an $m \times n$ matrix with entries in K^* , such that all its m -minors are non-zero. It exists because K is infinite. Consider the K -algebra map

$$\Phi : S = K[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n] \rightarrow K[y_1, \dots, y_n]$$

induced by $\Phi(x_{ij}) = a_{ij}y_j$ for every i, j . By construction the kernel of Φ is generated by $n(m-1)$ linear forms. Let $Y = \Phi(X) = (a_{ij}y_j)$. Denote by $[c_1, \dots, c_m]_W$ the minor with column indices c_1, \dots, c_m of an $m \times n$ matrix W . By construction

$$\Phi([c_1, \dots, c_m]_X) = [c_1, \dots, c_m]_A y_{c_1} \cdots y_{c_m}.$$

Hence, by our assumption on A , we have that

$$\Phi(I_m(X)) = I_m(Y) = (y_{c_1} \cdots y_{c_m} : 1 \leq c_1 < \cdots < c_m \leq n)$$

i.e., $I_m(Y)$ is generated by all the square-free monomials in y_1, \dots, y_n of total degree m . In particular it has codimension $n - m + 1$. It follows that $I_m(Y)$ is resolved by the Eagon-Northcott complex, hence $\mathrm{Ker} \Phi$ is generated by a regular sequence on $S/I_m(X)$. Now let \prec be any term order on S and let D be the ideal generated

by the leading terms of the maximal minors of X with respect to \prec . We have $D \subseteq \text{in}_\prec(I_m(X))$ and

$$\Phi(\text{in}_\prec([c_1, \dots, c_m]_X)) = \Phi(x_{\sigma_1 c_1} \cdots x_{\sigma_m c_m}) = a_{\sigma_1 c_1} \cdots a_{\sigma_m c_m} y_{c_1} \cdots y_{c_m}$$

for some $\sigma \in S_m$. Hence

$$\Phi(D) = I_m(Y).$$

We apply Lemma 1.2 to the following data:

$$M = S/I_m(X), \quad T = S/D, \quad N = S/\text{in}_\prec(I_m(X)) \text{ and } J = \text{Ker } \Phi$$

to conclude that $D = \text{in}_\prec(I_m(X))$, and the Betti numbers of $I_m(X)$ equals those of D . \square

Can one generalize Theorem 1.1 to ideals of maximal minors of matrices of linear forms? In Sections 3 and 4 we will give positive answers to the question by assuming the matrix is multigraded, either by rows or by columns. In general however one cannot expect too much, as the following remark shows.

Remark 1.3. One can consider various properties related to the existence of Gröbner bases and various families of matrices of linear forms. For instance we can look at the following properties for the ideal $I_m(L)$ of m -minors of an $m \times n$ matrix L of linear forms in a polynomial ring S :

- (a) $I_m(L)$ has a Gröbner basis of elements of degree m with respect to some term order and possibly after a change of coordinates.
- (b) $I_m(L)$ has a Gröbner basis of elements of degree m with respect to some term order and in the given coordinates.
- (c) Property (b) holds and the associated initial ideal has a linear resolution.
- (d) $I_m(L)$ has a universal Gröbner basis of elements of degree m .

We consider the following families of matrices of linear forms:

- (1) No further assumption on L is made.
- (2) $I_m(L)$ has codimension $n - m + 1$.
- (3) The entries of L are linearly independent over the base field (i.e., L arises from a matrix of variables by a change of coordinates).

What we know (and do not know) is summarized in the following table:

	(a)	(b)	(c)	(d)
(1)	no	no	no	no
(2)	yes	no	no	no
(3)	yes	?	?	no

There are ideals of 2-minors of 2×4 matrices of linear forms that define non-Koszul ring (see [7, Remark 3.6]). Hence those ideals cannot have a single Gröbner bases of quadrics (not even after a change of coordinates). This explains the four “no” in the first row of the table.

Every initial ideal of the ideal of 2-minors of

$$\begin{pmatrix} x_1 + x_2 & x_3 & x_3 \\ 0 & x_1 & x_2 \end{pmatrix}$$

has a generator in degree 3 if the characteristic of the base field is $\neq 2$. The codimension of $I_2(L)$ is 2. This example explains the three “no” in the second

row of the table. The “yes” in the second row follows because the generic initial ideal with respect to the reverse lexicographic order is generated in degree m under assumption (2).

Finally, the matrix

$$\begin{pmatrix} x_1 & x_4 & x_3 \\ x_5 & x_1 + x_6 & x_2 \end{pmatrix}$$

belongs to the family (3) and the initial ideal with respect to any term order satisfying $x_1 > x_2 > \cdots > x_6$ has a generator in degree 3. This explains the “no” in the third row. The “yes” is there because (3) is contained in (2).

It remains open whether the ideal of maximal minors of a matrix in the family (3) has at least a Gröbner basis of elements of degree m in the given coordinates, and whether the associated initial ideal has a linear resolution.

2. RADICAL AND BOREL FIXED IDEALS

The goal of the section is to prove Theorem 2.5, a rigidity result for multigraded Hilbert series associated to radical multigraded Borel fixed ideals. Special cases of it appeared already in [5] and [1]. We will introduce the geometric multidegree, a generalization of the notion of multidegree of Miller and Sturmfels [11, Chap.8], that allows us to deal with minimal components of various codimensions in the case of Borel fixed ideals.

Given $m \in \mathbb{N}$ and $(n_1, \dots, n_m) \in \mathbb{N}^m$ let S be the polynomial ring in the set of variables x_{ij} with $1 \leq i \leq m$ and $1 \leq j \leq n_i$ over an infinite field K , with grading induced by $\deg(x_{ij}) = e_i \in \mathbb{Z}^m$. Let M be a finitely generated, \mathbb{Z}^m -graded S -module. The multigraded Hilbert series of M is:

$$\text{HS}(M, y) = \text{HS}(M, y_1, \dots, y_m) = \sum_{a \in \mathbb{Z}^m} (\dim M_a) y^a \in \mathbb{Q}[[y_1, \dots, y_m]][y_1^{-1}, \dots, y_m^{-1}].$$

The \mathcal{K} -polynomial of M is:

$$\mathcal{K}(M, y) = \mathcal{K}(M, y_1, \dots, y_m) = \prod_{i=1}^m (1 - y_i)^{n_i} \text{HS}(M, y).$$

Indeed,

$$\mathcal{K}(M, y) \in \mathbb{Z}[y_1, \dots, y_m][y_1^{-1}, \dots, y_m^{-1}].$$

The group $G = \text{GL}_{n_1}(K) \times \cdots \times \text{GL}_{n_m}(K)$ acts on S as the group of \mathbb{Z}^m -graded K -algebras automorphisms. Let $B = B_{n_1}(K) \times \cdots \times B_{n_m}(K)$ be the Borel subgroup of G consisting of the upper triangular matrices with arbitrary non-zero diagonal entries. An ideal I is said to be Borel fixed if $g(I) = I$ for every $g \in B$. Borel fixed ideals are monomial ideals that can be characterized in a combinatorial way by means of exchange properties as it is explained in [8, Thm. 15.23]. Indeed in [8, Thm. 15.23] details are given in the standard graded setting but, as observed in [2, Sect.1], the same characterization holds also in the multigraded setting. Given a term order \prec such that $x_{ik} \prec x_{ij}$ for $j > k$, one can associate a (multigraded) generic initial ideal $\text{gin}_{\prec}(I)$ to any \mathbb{Z}^m -graded ideal I of S . $\text{gin}_{\prec}(I)$ is Borel fixed.

The prime Borel fixed ideals are easy to describe. Set

$$U = \{(b_1, \dots, b_m) \in \mathbb{N}^m : b_i \leq n_i \text{ for every } i = 1, \dots, m\}.$$

The following assertion follows immediately from the definition.

Lemma 2.1. *For every vector $b \in U$ the ideal*

$$P_b = (x_{ij} : i = 1, \dots, m \text{ and } 1 \leq j \leq b_i)$$

is prime and Borel fixed, and every prime Borel fixed ideal is of this form.

Lemma 2.2. *The associated prime ideals of a Borel fixed ideal I are Borel fixed.*

Proof. Let P be an associated prime to S/I . Clearly P is monomial (i.e., generated by variables) because I is monomial. We have to prove that if $x_{ij} \in P$ then also $x_{ik} \in P$ for all $k < j$. We may write $P = I : f$ for some monomial f . Let α be the exponent of x_{ij} in f . Consider $g \in B$ such that $g(x_{ij}) = x_{ij} + x_{ik}$ and fixes all the other variables. Then $g(x_{ij}f) \in I$ because $x_{ij}f \in I$. The monomial $x_{ik}^{\alpha+1}f/x_{ij}^\alpha$ appears with nonzero coefficient in $g(x_{ij}f)$. Hence $x_{ik}^{\alpha+1}f/x_{ij}^\alpha \in I$ and $x_{ik}^{\alpha+1}f \in I$. In other words, $x_{ik}^{\alpha+1} \in I : f = P$ and hence $x_{ik} \in P$. \square

Lemma 2.3. *Let I be a radical and Borel fixed ideal. Then every minimal generator of I has multidegree bounded above by $(1, 1, \dots, 1) \in \mathbb{Z}^m$.*

Proof. Consider a generator f of I of degree (a_1, \dots, a_m) . We may write $f = ug$ with u a monomial of degree a_1e_1 . Since I is Borel fixed we have $x_{1j}^{a_1}g \in I$, where $j = \min\{k : x_{1k}|u\}$. Since I is radical we have $x_{1j}g \in I$, and $x_{1j}g$ is a proper divisor of f unless $a_1 = 1$. \square

Lemma 2.4. *Let I be a radical Borel fixed ideal and let $\{P_{b_1}, \dots, P_{b_c}\}$, with $b_1, \dots, b_c \in U$, be the minimal primes of I . Then I is the Alexander dual of the polarization of*

$$J = \left(\prod_{b_{ij} > 0} x_j^{b_{ij}} : i = 1, \dots, c \right) \subset K[x_1, \dots, x_m].$$

In particular, if all the generators of I have the same multidegree, then I has a linear resolution.

Proof. The first assertion follows immediately from the definition of polarization and Alexander duality, see [11, Chap.5]. For the second, one observes that if all the generators of I have degree, say, $e_1 + e_2 + \dots + e_u \in \mathbb{Z}^m$, then I is the Alexander dual of the polarization of an ideal $J \subset K[x_1, \dots, x_m]$ involving only variables x_i with $i \leq u$ and whose radical is (x_1, \dots, x_u) . Hence J defines a Cohen-Macaulay ring, and so does its polarization. Finally one applies the Eagon-Reiner Theorem [9, Thm. 8.1.9]. \square

The goal of this section is to prove the following:

Theorem 2.5. *Let $I, J \subset S$ be Borel fixed ideals such that $\text{HS}(I, y) = \text{HS}(J, y)$. If I is radical then $I = J$.*

The most important consequence of Theorem 2.5 is the following rigidity result:

Corollary 2.6. *Let I be a radical Borel fixed ideal. For every multigraded ideal J with $\text{HS}(J, y) = \text{HS}(I, y)$ one has:*

- (a) $\text{gin}_\prec(J) = I$ for every term order \prec .
- (b) J is radical.
- (c) J has a linear resolution whenever I has a linear resolution.
- (d) S/J is Cohen-Macaulay whenever S/I is Cohen-Macaulay.

- (e) $\beta_{i,a}(S/J) \leq \beta_{i,a}(S/I)$ for every $i \in \mathbb{N}$ and $a \in \mathbb{Z}^m$ and $\beta_{i,a}(S/J) = 0$ if $a \not\leq (1, 1, \dots, 1) \in \mathbb{Z}^m$.

Proof. The ideal $\text{gin}_{\prec}(J)$ is a Borel fixed ideal and $\text{HS}(J, y) = \text{HS}(\text{gin}_{\prec}(J), y)$. Since, by assumption, $\text{HS}(J, y) = \text{HS}(I, y)$, we may conclude, by virtue of Theorem 2.5 that $\text{gin}_{\prec}(J) = I$. This proves (a). Statements (b), (c) and (d) are standard applications of well-known principles. Finally (e) follows from Lemma 2.3 and from the bounds derived from the Taylor complex, see [11, Chap.6] \square

In order to prove Theorem 2.5 we need the following definition.

Definition 2.7. For every finitely generated \mathbb{Z}^m -graded S -module M we set

$$\mathcal{C}(M, y) = \mathcal{K}(M, 1 - y_1, \dots, 1 - y_m) \in \mathbb{Z}[[y_1, \dots, y_m]]$$

and we define the G-multidegree (geometric multidegree) of M as

$$\mathcal{G}(M, y) = \sum c_a y^a \in \mathbb{Z}[y_1, \dots, y_m]$$

where the sum is over the $a \in \mathbb{Z}^m$ which are minimal in the support of $\mathcal{C}(M, y)$ and c_a is the coefficient of y^a in $\mathcal{C}(M, y)$.

The following result follows immediately from the definition above.

Proposition 2.8. (1) Let P be a prime ideal generated by variables and let $a(P)$ be the vector whose i -th coordinate is $\#(P \cap \{x_{i1}, \dots, x_{in_i}\})$. Then

$$\mathcal{G}(S/P, y) = y^{a(P)}$$

- (2) One has $a(P_b) = b$ for every $b \in U$ and for $b_1, b_2 \in U$ one has $P_{b_1} \subseteq P_{b_2}$ if and only if $y^{b_1} | y^{b_2}$.

The key observation is the following:

Proposition 2.9. Let I be a Borel fixed ideal. One has

$$\mathcal{G}(S/I, y) = \sum_{i=1}^c \text{length}((S/I)_{P_{b_i}}) y^{b_i}$$

where $\text{Min}(I) = \{P_{b_1}, \dots, P_{b_c}\}$ for some $b_i \in U$.

Proof. In order to compute the \mathcal{K} -polynomial of $M = S/I$, consider a filtration of \mathbb{Z}^m -graded modules

$$0 = M_0 \subset M_1 \subset \dots \subset M_h = M$$

such that $M_i/M_{i-1} \simeq S/P_i(-v_i)$. Here P_i is a \mathbb{Z}^m -graded monomial prime ideal and $v_i = (v_{i1}, \dots, v_{im}) \in \mathbb{Z}^m$. Existence of such a filtration follows from basic commutative algebra facts, see [8, Prop.3.7]. Furthermore

$$\text{Min}(I) \subseteq \text{Ass}(S/I) \subseteq \{P_1, \dots, P_h\}.$$

Hence we have

$$\mathcal{K}(S/I, y) = \sum_{i=1}^h \mathcal{K}(S/P_i(-v_i), y) = \sum_{i=1}^h y^{v_i} \mathcal{K}(S/P_i, y).$$

It follows that

$$\mathcal{C}(S/I, y) = \sum_{i=1}^h \prod_{j=1}^m (1 - y_j)^{v_{ij}} \mathcal{C}(S/P_i, y).$$

Then the support of the polynomial $\prod_{j=1}^m (1 - y_j)^{v_{ij}} \mathcal{C}(S/P_i, y)$ contains exactly one minimal element, namely $y^{a(P_i)}$, which appears in the polynomial with coefficient 1. It follows that $\mathcal{G}(S/I, y)$ is obtained as the sum of the terms which are minimal in the support of the polynomial

$$(2.9.1) \quad \sum_{i=1}^h y^{a(P_i)}$$

Now the elements that are minimal support in the support of (2.9.1) are exactly the y^{b_i} corresponding to the minimal primes P_{b_i} . This follows from Proposition 2.8, since if $P \subseteq P'$, then $y^{a(P)} | y^{a(P')}$. Finally, by standard localization arguments we have that each minimal prime P_{b_i} appears in the multiset $\{P_1, \dots, P_h\}$ as many times as $\text{length}((S/I)_{P_{b_i}})$. \square

We are finally ready to prove Theorem 2.5.

Proof of Theorem 2.5. Since I and J have the same Hilbert series we have that $\mathcal{C}(S/I, y) = \mathcal{C}(S/J, y)$ and hence

$$\mathcal{G}(S/I, y) = \mathcal{G}(S/J, y).$$

It follows by Proposition 2.9 that $\text{Min}(I) = \text{Min}(J)$. Since I is radical, the coefficients in $\mathcal{G}(S/I, y)$ are all 1. Hence the primary decomposition of J is of the form $I \cap Q$, where Q is the intersection of the components associated to the embedded prime ideals of J , if any. We deduce that $J \subseteq I$ and the Hilbert series forces the equality $I = J$. \square

3. COLUMN-GRADED IDEALS OF MAXIMAL MINORS

Consider $S = K[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$ graded by $\deg(x_{ij}) = e_j \in \mathbb{Z}^n$. Let $L = (L_{ij})$ be a $m \times n$ matrix of linear forms which is column-graded, that is, whose entries L_{ij} satisfy $\deg(L_{ij}) = e_j$. In other words,

$$L_{ij} = \sum_{k=1}^m \lambda_{ijk} x_{kj}$$

where $\lambda_{ijk} \in K$. As a first direct application of Corollary 2.6 we have:

Theorem 3.1. *Let $L = (L_{ij})$ be a $m \times n$ matrix which is column-graded and assume that the codimension of $I_m(L)$ is $n - m + 1$. Then $I_m(L)$ is radical and the maximal minors of L form a universal Gröbner basis of it. Furthermore every initial ideal of $I_m(L)$ is radical, has a linear resolution, and its Betti numbers equals those of $I_m(L)$.*

Proof. We may assume without loss of generality that K is infinite. Let $I = (x_{1j_1} x_{1j_2} \cdots x_{1j_m} : 1 \leq j_1 < j_2 < \cdots < j_m \leq n)$. Then I is generated by the maximal minors of a column-graded matrix whose (i, j) -th entry is $a_{ij} x_{1j}$ with randomly chosen scalars a_{ij} . Since the codimension of I is $n - m + 1$, by the Eagon-Northcott complex it follows that I and $I_m(L)$ have the same multigraded Hilbert series. Since I is radical and Borel fixed, we may apply Corollary 2.6 with $J = I_m(L)$ or J equal any initial ideal of $I_m(L)$ to conclude. \square

We want now to generalize Theorem 3.1 and get rid of the assumption on the codimension of $I_m(L)$.

Theorem 3.2. *Let $L = (L_{ij})$ be an $m \times n$ matrix which is column-graded. Then:*

- (a) *The maximal minors of L form a universal Gröbner basis of $I_m(L)$.*
- (b) *$I_m(L)$ is radical and it has a linear resolution.*
- (c) *Any initial ideal J of $I_m(L)$ is radical and has a linear resolution. In particular, $\beta_{i,j}(I_m(L)) = \beta_{i,j}(J)$ for all i, j .*
- (d) *Assume that $I_m(L) \neq 0$ and that no column of $I_m(L)$ is identically 0. Then the projective dimension of $I_m(L)$ (and hence of all its initial ideals) is $n - m$.*

Proof. Again we may assume that K is infinite. Fix a term order \prec . It is not restrictive to assume that $x_{1j} \succ x_{ij}$ for all $i \neq 1$ and j ; set for simplicity $x_j = x_{1j}$. Let

$$I = (x_{j_1} \cdots x_{j_m} \mid [j_1, \dots, j_m]_L \neq 0).$$

We claim that $I = \text{gin}_{\prec}(I_m(L))$. First we note that $I \subseteq \text{gin}_{\prec}(I_m(L))$. This is because if $[j_1, \dots, j_m]_L \neq 0$, then $I_m(L)$ contains a non-zero element of degree $e_{j_1} + \cdots + e_{j_m}$ and its initial term in generic coordinates is $x_{j_1} \cdots x_{j_m}$.

Next note that I is the Stanley-Reisner ideal of the Alexander dual of the matroid dual M_L^* of the matroid M_L associated to L . As such, I has a linear resolution by the Eagon-Reiner Theorem [9, Thm.8.1.9], since M_L^* is Cohen-Macaulay. By Buchberger's Algorithm, in order to prove that $I = \text{gin}_{\prec}(I_m(L))$ it suffices to show that any S -pair associated to a linear syzygy among the generators of I reduces to 0. Any such linear syzygy involves at most $m + 1$ column indices in total. After renaming the column indices, we may assume that the syzygy in question involves the column indices $\{1, 2, \dots, m + 1\}$. Set

$$d = e_1 + e_2 + \cdots + e_{m+1}.$$

To prove that the S -polynomial reduces to 0 we may as well prove that $\dim I_m(L)_d \leq \dim I_d$. Let

$$W = \{u : 1 \leq u \leq m + 1 \text{ and } [\{1, \dots, m + 1\} \setminus \{u\}]_L \neq 0\}.$$

Renaming if needed, we may assume that

$$W = \{1, 2, \dots, s\}$$

By definition I_d is generated by the set of monomials

$$\left\{ \frac{x_1 x_2 \cdots x_{m+1}}{x_j} x_{ij} : j = 1, \dots, s \text{ and } i = 1, \dots, m \right\}$$

whose cardinality is easily seen to be $sm - s + 1$. Hence it remains to prove that

$$\dim I_m(L)_d \leq sm - s + 1.$$

Denote by Ω the first syzygy module of $\{[\{1, \dots, m + 1\} \setminus \{u\}]_L : u = 1, \dots, s\}$. Since

$$\dim I_m(L)_d = sm - \dim \Omega_d$$

it suffices to show that

$$\dim \Omega_d \geq s - 1.$$

Let L_1 be the submatrix of L consisting of the first s columns of L . Since the rows of L_1 are elements of Ω_d , it is enough to show that L_1 has at least $s - 1$ linearly independent rows over K . By contradiction, if this is not the case, by applying invertible K -linear operations to the rows of L we may assume that the last

$m-s+2$ rows of L_1 are identically zero. In particular the minor $[2, \dots, m+1]_L = 0$, contradicting our assumptions.

Since I is Borel fixed and radical with $\text{HS}(I, y) = \text{HS}(I_m(L), y)$, we may apply Corollary 2.6 and deduce (a), (b) and (c). For (d) one observes that, under the assumption that no column of L is 0 and $I_m(L) \neq 0$, the ideal I is non-zero and each of the variables x_1, \dots, x_n is involved in some generator. Then M_L^* has dimension $n-m$ and has no cone-points. This implies that the Stanley-Reisner ring of M_L^* has regularity $n-m$, as it is 2-Cohen-Macaulay (see [13, pg.94] for details). By [9, Prop.8.1.10] the projective dimension of I (that is the Alexander dual of M_L^*) is $n-m$. \square

4. ROW-GRADED IDEALS OF MAXIMAL MINORS

In this section we treat ideals of maximal minors of row-graded matrices. Consider $S = K[x_{ij} : i = 1, \dots, m \text{ and } j = 1, \dots, n]$ graded by $\deg(x_{ij}) = e_i \in \mathbb{Z}^m$. Let $L = (L_{ij})$ be a $m \times n$ matrix of linear forms which is row-graded, i.e., whose entries L_{ij} satisfy $\deg(L_{ij}) = e_i$. In other words,

$$L_{ij} = \sum_{k=1}^m \lambda_{ijk} x_{ik}$$

where $\lambda_{ijk} \in K$. Observe that in the row-graded case we cannot expect that the maximal minors of X form a Gröbner basis simply because every maximal minor might have the same leading term. Nevertheless we can prove the following:

Theorem 4.1. *Let $L = (L_{ij})$ be an $m \times n$ matrix which is row-graded and assume that the codimension of $I_m(L)$ is $n-m+1$. Then $I_m(L)$ is radical and every initial ideal is generated by elements of total degree m (equivalently, there is a universal Gröbner basis of elements of degree m). Furthermore every initial ideal of $I_m(L)$ is radical, has a linear resolution, and its Betti numbers equals those of $I_m(L)$.*

Set

$$I = (x_{1j_1} \cdots x_{mj_m} : j_1 + \cdots + j_m \leq n).$$

Theorem 4.1 follows immediately from Corollary 2.6 and from the following proposition, by observing that I is radical and Borel fixed. Notice that Corollary 2.6 also implies that $I = \text{gin}_{\prec}(I_m(L))$ for every term order \prec .

Proposition 4.2. *Under the assumptions of Theorem 4.1, the \mathbb{Z}^m -graded Hilbert series of $I_m(L)$ equals that of I .*

Proof. The Hilbert series of $I_m(L)$ equals that of $I_m(X)$ with $X = (x_{ij})$, because both ideals are resolved by the multigraded version of the Eagon-Northcott complex. Hence we may assume without loss of generality that $L = X$. We will show that $S/I_m(X)$ and S/I have the same \mathcal{K} -polynomial.

Let $\mathcal{K}_{m,n}(y)$ be the \mathcal{K} -polynomial of $S/I_m(X)$. By looking at the diagonal initial ideal of $I_m(X)$ one obtains the recursion:

$$\mathcal{K}_{m,n}(y) = (1 - y_m)\mathcal{K}_{m,n-1}(y_1, \dots, y_m) + y_m\mathcal{K}_{m-1,n-1}(y_1, \dots, y_{m-1}).$$

Solving the recursion or, alternatively, by looking directly at the multigraded version of the Eagon-Northcott complex, one obtains:

$$(4.2.1) \quad \mathcal{K}_{m,n}(y) = 1 - \left(\prod_{i=1}^m y_i \right) \sum_{k=0}^{n-m} (-1)^k \binom{n}{m+k} h_k(y_1, \dots, y_m)$$

where $h_k(y_1, \dots, y_m)$ is the complete symmetric polynomial of degree k , i.e., the sum of all the monomials of degree k in the variables y_1, \dots, y_m .

We now compute the \mathcal{K} -polynomial of S/I . For $b \in [n]^m$ set $x_b = x_{1b_1} x_{2b_2} \cdots x_{mb_m}$ so that

$$I = (x_b : b \in \mathbb{N}_{>0}^m \text{ and } |b| \leq n).$$

Extend the natural partial order, i.e. $x_b \leq x_c$ if $b \leq c$ coefficientwise, to a total order $<$ (no matter how). For every $b \in [n]^m$ we have:

$$(4.2.2) \quad (x_c : x_c < x_b) : x_b = (x_{ij} : i = 1, \dots, m \text{ and } 1 \leq j < b_i).$$

Filtering I according to $<$ and using (4.2.2) one obtains:

$$(4.2.3) \quad \mathcal{K}(S/I, y) = 1 - y_1 \cdots y_m \sum_b \prod_{i=1}^m (1 - y_i)^{b_i - 1}$$

where the sum \sum_b is over all the $b \in \mathbb{N}_{>0}^m$ and $|b| \leq n$. Setting $c = b - (1, \dots, 1)$ and replacing b with c in (4.2.3) we obtain:

$$(4.2.4) \quad \mathcal{K}(S/I, y) = 1 - y_1 \cdots y_m \sum_c \prod_{i=1}^m (1 - y_i)^{c_i}$$

where the sum \sum_c is over all the $c \in \mathbb{N}^m$ and $|c| \leq n - m$. We may rewrite the last expression as:

$$(4.2.5) \quad \mathcal{K}(S/I, y) = 1 - y_1 \cdots y_m \sum_{k=0}^{n-m} h_k(1 - y_1, \dots, 1 - y_m).$$

Taking into consideration (4.2.1) and (4.2.5), it remains to prove that:

$$(4.2.6) \quad \sum_{k=0}^{n-m} h_k(1 - y_1, \dots, 1 - y_m) = \sum_{k=0}^{n-m} (-1)^k \binom{n}{m+k} h_k(y_1, \dots, y_m)$$

or equivalently, by replacing y_i with $-y_i$ in (4.2.6), it is left to show that:

$$(4.2.7) \quad \sum_{k=0}^{n-m} h_k(1 + y_1, \dots, 1 + y_m) = \sum_{k=0}^{n-m} \binom{n}{m+k} h_k(y_1, \dots, y_m).$$

Setting $t = n - m$, (4.2.7) is equivalent to the assertion that the equality:

$$(4.2.8) \quad \sum_{k=0}^t h_k(1 + y_1, \dots, 1 + y_m) = \sum_{k=0}^t \binom{m+t}{m+k} h_k(y_1, \dots, y_m)$$

holds for every m and t . The formula (4.2.8) can be derived from the more precise:

$$(4.2.9) \quad h_t(1 + y_1, \dots, 1 + y_m) = \sum_{k=0}^t \binom{m+t-1}{m+k-1} h_k(y_1, \dots, y_m).$$

Equation (4.2.9) can be proved by (long and tedious) induction on m . The following simple argument using generating functions was suggested by Christian Krattenthaler. First notice that:

$$(4.2.10) \quad \sum_{t \geq 0} h_t(y_1, \dots, y_m) z^t = \prod_{i=1}^m \frac{1}{1 - y_i z}.$$

Replacing in (4.2.10) y_i with $y_i + 1$ and observing that

$$\prod_{i=1}^m \frac{1}{1 - (y_i + 1)z} = \frac{1}{(1 - z)^m} \prod_{i=1}^m \frac{1}{1 - y_i \frac{z}{1-z}}$$

we have:

$$(4.2.11) \quad \sum_{t \geq 0} h_t(1 + y_1, \dots, 1 + y_m) z^t = \sum_{t \geq 0} h_t(y_1, \dots, y_m) \frac{z^t}{(1 - z)^{t+m}}.$$

Expanding the right-hand side of (4.2.11) one obtains (4.2.9). \square

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